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Semistable divisorial contractions

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Abstract

The semistable minimal model program is a special case of the minimal model program concerning 3-folds fibred over a curve and birational morphisms preserving this structure. We classify semistable divisorial contractions which contract the exceptional divisor to a normal point of a fibre. Our results can be applied to describe compact moduli spaces of surfaces.

1. Introduction

The minimal model program is a generalisation of the classical theory of minimal models of surfaces. Given a 3-fold X, the minimal model program constructs a birational model Y such that either K_Y is non-negative, or there is a fibration $Y \to S$ with K_Y negative on the fibres. The 3-fold Y is obtained via a sequence of elementary birational maps called divisorial contractions and flips. A divisorial contraction is a birational morphism $\phi : X \to X'$ which contracts an irreducible divisor $E \subset X$ to a point or curve on X'. A flip is a birational map $\phi : X \dashrightarrow X'$ of the form $\phi = g^{-1} \circ f$, where $f : X \to Z$ and $g : X' \to Z$ are birational morphisms which contract bunches of curves to points of Z.

Suppose given a 3-fold X fibred over a curve T such that the total space and the fibres have appropriately mild singularities (including, e.g., the case that X is smooth and the fibres are simple normal crossing divisors). We can run a relative minimal model program for the family X/T, contracting only curves which lie in the fibres, so that each elementary birational map is defined over T. This process is called the semistable minimal model program [12, Chapter 7], and we say a contraction is semistable if it occurs in this context.

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In this paper we classify semistable divisorial contractions which contract the exceptional divisor to a normal point of a fibre.

The semistable minimal model program is an important tool in the construction and explicit description of compact moduli spaces of surfaces. For, given a family $\mathcal{X}^\times/T^\times$ of surfaces of general type over a punctured curve $T^\times = T\backslash\{0\}$, we can construct a canonical completion to a family $\mathcal{X}/T$ using the semistable minimal model program [13]. If M is a moduli space of surfaces of general type, there is a compactification $\overline{M}$ of M with boundary points corresponding to surfaces obtained as the limit of a family $\mathcal{X}^\times/T^\times$ of surfaces from M by the above process. More generally, one can compactify moduli spaces of surface-divisor pairs (X, D) such that $K_X + D$ is ample [1]. For example, the surface may be a K3 or abelian surface. In this case we construct limits using the log minimal model program, where the rôle of K_X is played by $K_X + D$. Our results also apply in the log case (for details, see Proposition 2.6). In fact, I have already used the results of this paper to give an explicit description of the degenerate surfaces which occur at the boundary of a compactification of a moduli space M of pairs [2,3]. Here M is the moduli space of pairs consisting of the plane together with a smooth curve of fixed degree $d \geqslant 4$; in other words M is the moduli space of smooth plane curves of degree d. Similarly, one can apply our methods to explain and extend the results of Hassett on stable reduction of plane curve singularities [5].

Since this article first appeared as a preprint [4], the classification of all divisorial contractions with centre a point has been completed [6–11]. However, the classification of the semistable contractions is much clearer—there are several exceptional cases which do not occur in the semistable context, and the remaining cases are organised into easily described families. Moreover, our proof of the classification is concise and instructive. Hence, in view of the applications to moduli problems, our results are of independent interest.

2. The classification

Notation 2.1. We refer to [12] for Mori theory background, including the definitions of the various classes of singularities we consider. In this paper log terminal means purely log terminal. We write $0 \in T$ for the analytic germ of a smooth curve. We use script letters to denote flat families of surfaces over T and regular letters for the special fibre, e.g.,

$$\begin{array}{ccc} X & \subset & \mathcal{X} \\ \downarrow & & \downarrow \\ 0 & \in & T \end{array}$$

Definition 2.2. We say a 3-fold $\mathcal{X}/(0 \in T)$ is *semistable terminal* if $(\mathcal{X}, X)$ is divisorial log terminal and $\mathcal{X}$ is terminal.

Definition 2.3. A *semistable divisorial contraction* is a birational morphism $\pi : \mathcal{Y} \to \mathcal{X}/T$ where

(1) $\mathcal{Y}/T$ and $\mathcal{X}/T$ are semistable terminal.
(2) The divisor $-K_{\mathcal{Y}}$ is π-ample.
(3) The relative Picard number $\rho(\mathcal{Y}/\mathcal{X}) = 1$.
(4) The exceptional locus of π is a divisor.

We recall the classification of semistable terminal singularities (cf. [13]). It can also be easily derived using the method of toric blowups explained in [14].

Theorem 2.4. *Let $(P \in \mathcal{X})/(0 \in T)$ be a germ of a semistable terminal 3-fold. Then, up to analytic isomorphisms of the 3-fold $\mathcal{X}$ and curve T, the family $\mathcal{X}/T$ is of one of the following forms*:

(1) $((xyz = t) \subset \mathbb{C}^4)/\mathbb{C}^1_t$.
(2) $((xy + tg(z^n, t) = 0) \subset \frac{1}{n}(1, -1, a, 0))/\mathbb{C}^1_t$, *where* $(a, n) = 1$.
(3) $((xy - z^{kn} + tg(z^n, t) = 0) \subset \frac{1}{n}(1, -1, a, 0))/\mathbb{C}^1_t$, *where* $(a, n) = 1$.
(4) $((f(x, y, z) + tg(x, y, z, t) = 0) \subset \mathbb{C}^4)/\mathbb{C}^1_t$, *where* $((f = 0) \subset \mathbb{C}^3)$ *is a Du Val singularity of type D or E.*

Here n is the index of $P \in \mathcal{X}$ (and $n = 1$ in cases (1) *and* (4)). *Conversely, any family $(P \in \mathcal{X})/(0 \in T)$ of this form such that the singularity $P \in \mathcal{X}$ is isolated is semistable terminal.*

Note that the special fibre X is normal only in cases (3) and (4). In case (3) the singularity $P \in X$ is a cyclic quotient singularity:

$$X = \left(\left(xy - z^{kn} = 0\right) \subset \frac{1}{n}(1, -1, a)\right) \cong \mathbb{A}^2_{u,v} \Big/ \frac{1}{kn^2}(1, kna - 1)$$

where $x = u^{kn}$, $y = v^{kn}$ and $z = uv$. These are the cyclic quotient singularities of class T [13, Definition 3.7]. If $n = 1$ then X is a Du Val singularity of type A or smooth. In case (4) the singularity $P \in X$ is a Du Val singularity of type D or E.

We can now state our result:

Theorem 2.5. *Let $\pi : (E \subset \mathcal{Y}) \to (P \in \mathcal{X})/T$ be a semistable divisorial contraction such that the special fibre of $\mathcal{X}/T$ is normal. Then there is an analytic isomorphism*

$$P \in \mathcal{X}/T \cong \left(\left(f(x, y, z) + tg(x, y, z, t) = 0\right) \subset \frac{1}{n}(1, -1, a, 0)\right) \Big/ \mathbb{C}^1_t$$

such that π is given by the weighted blowup of x, y, z, t with weights $w = (w_0, 1)$, where $w(tg) \geqslant w(f)$. Here the function $f + tg$ is μ_n-invariant and $(a, n) = 1$. We have the following possibilities for n, f and w_0:

(1) $n \in \mathbb{N}$ *arbitrary*:

(T) $f(x, y, z) = xy - z^{kn}$ *and* w_0 *is a primitive vector in the lattice* $\mathbb{Z}^3 + \mathbb{Z}\frac{1}{n}(1, -1, a)$ *such that* f *is homogeneous with respect to the weights* w_0 *of* x, y, z.

(2) $n = 1$:

(D_m) $f(x, y, z) = x^2 + y^2 z + z^{m-1}$ *and* $w_0 = (m-1, m-2, 2)$*, some* $m \geqslant 4$.

(E_6) $f(x, y, z) = x^2 + y^3 + z^4$ *and* $w_0 = (6, 4, 3)$.

(E_7) $f(x, y, z) = x^2 + y^3 + yz^3$ *and* $w_0 = (9, 6, 4)$.

(E_8) $f(x, y, z) = x^2 + y^3 + z^5$ *and* $w_0 = (15, 10, 6)$.

Conversely, let $P \in \mathcal{X}/T \cong (f + tg = 0) \subset \frac{1}{n}(1, -1, a, 0)/\mathbb{C}^1_t$ *and let* $\pi : \mathcal{Y} \to \mathcal{X}$ *be the birational map induced by the weighted blowup of* x, y, z, t *with weights* $w = (w_0, 1)$*, where* n*,* f *and* w_0 *are as above and* g *is chosen so that* $\mathcal{X}$ *has an isolated singularity and* $w(tg) \geqslant w(f)$*. Then the families* $\mathcal{Y}/T$ *and* $\mathcal{X}/T$ *are semistable terminal, the divisor* $-K_{\mathcal{Y}}$ *is* π*-ample and the exceptional locus of* π *is an irreducible divisor. Thus* π *is a semistable divisorial contraction if* $\rho(\mathcal{Y}/\mathcal{X}) = 1$ (*which is automatic if* $\mathcal{X}$ *is* $\mathbb{Q}$*-factorial at* P).

Finally, we describe how our results apply to the log case. Our result and notation below are not used in the remainder of this paper. We say a pair $(\mathcal{X}, \mathcal{D})/(0 \in T)$ is *semistable log terminal* if $(\mathcal{X}, \mathcal{D} + X)$ is divisorial log terminal. A *semistable log divisorial contraction* is a birational morphism $\pi : (\mathcal{Y}, \mathcal{D}_{\mathcal{Y}}) \to (\mathcal{X}, \mathcal{D})/T$ of semistable log terminal pairs such that $-(K_{\mathcal{Y}} + \mathcal{D}_{\mathcal{Y}})$ is π-ample, the exceptional locus is a divisor, and $\rho(\mathcal{Y}/\mathcal{X}) = 1$.

Proposition 2.6. *Let* $\pi : (\mathcal{Y}, \mathcal{D}_{\mathcal{Y}}) \to (\mathcal{X}, \mathcal{D})/T$ *be a semistable log divisorial contraction which contracts the exceptional divisor to a point on the special fibre* X*. Assume that the divisor* $\mathcal{D}$ *is* $\mathbb{Q}$*-Cartier and the general fibre of* $\mathcal{X}/T$ *is smooth. Then* $\pi : \mathcal{Y} \to \mathcal{X}/T$ *is a semistable divisorial contraction.*

Proof. Our assumptions imply that $\mathcal{Y}/T$ and $\mathcal{X}/T$ are semistable terminal. It remains to show that $-K_{\mathcal{Y}}$ is π-ample. Let Γ be a general curve in the exceptional divisor E. Then $(K_{\mathcal{Y}} + \mathcal{D}_{\mathcal{Y}}) \cdot \Gamma < 0$ by assumption and $\mathcal{D}_{\mathcal{Y}} \cdot \Gamma \geqslant 0$ since Γ is not contained in $\mathcal{D}_{\mathcal{Y}}$. Thus $K_{\mathcal{Y}} \cdot \Gamma < 0$ as required. □

3. Proof of the classification

Proposition–Definition 3.1. *Let* $\pi : \mathcal{Y} \to \mathcal{X}/T$ *be a semistable divisorial contraction with exceptional divisor* E*. Assume that* E *is contracted to a point* $P \in \mathcal{X}$ *and that* X *is normal. Write* $Y = Y_1 + E$*, where* Y_1 *is the strict transform of* X *and* $F = E|_{Y_1}$*. Then the map* $p : Y_1 \to X$ *satisfies the following conditions*:

(1) *The surface* X *is log terminal and the pair* (Y_1, F) *is log terminal.*
(2) *The divisor* $-(K_{Y_1} + F)$ *is* p*-ample.*
(3) *The relative Picard number* $\rho(Y_1/X)$ *equals* 1.
(4) *The exceptional locus of* p *is* F*, a divisor.*

We say a map $p : Y_1 \to X$ *satisfying the conditions above is a log divisorial contraction.*

Proof. The pair $(\mathcal{X}, X)$ is dlt and X is normal by assumption, hence X is log terminal by adjunction. We have $(K_{\mathcal{Y}} + Y)|_{Y_1} = K_{Y_1} + F$ by adjunction. The pair $(\mathcal{Y}, Y)$ is dlt, so F is smooth and irreducible and (Y_1, F) is log terminal. The divisor $-K_{\mathcal{Y}}$ is π-ample thus the restriction $-(K_{Y_1} + F)$ is p-ample. □

Theorem 3.2. *Let $P \in X$ be a log terminal surface singularity which admits a $\mathbb{Q}$-Gorenstein smoothing* (*equivalently, X occurs as the special fibre of a semistable terminal family $\mathcal{X}/T$*). *Then the log divisorial contractions $p:(E \subset Y_1) \to (P \in X)$ are precisely the following*:

(1) *There is an isomorphism $P \in X \cong \mathbb{A}^2_{u,v}/\frac{1}{kn^2}(1, kna - 1)$, where $(a, n) = 1$, such that p is given by the weighted blowup of u, v with weights α. Here α is a primitive vector in the lattice $N = \mathbb{Z}^2 + \mathbb{Z}\frac{1}{kn^2}(1, kna - 1)$.*
Equivalently, there is an isomorphism $P \in X \cong (xy - z^{kn} = 0) \subset \frac{1}{n}(1, -1, a)$ such that p is given by the weighted blowup of x, y, z with weights w_0. Here w_0 is a primitive vector in the lattice $\mathbb{Z}^3 + \mathbb{Z}\frac{1}{n}(1, -1, a)$ such that f is homogeneous with respect to these weights.
(2) *The singularity $P \in X$ is a Du Val singularity of type D or E. Then $P \in X \cong ((f(x, y, z) = 0) \subset \mathbb{C}^3)$ and p is given by the weighted blowup of x, y, z with weights w_0, where*
(D_m) *$f(x, y, z) = x^2 + y^2z + z^{m-1}$ and $w_0 = (m - 1, m - 2, 2)$, some $m \geqslant 4$.*
(E_6) *$f(x, y, z) = x^2 + y^3 + z^4$ and $w_0 = (6, 4, 3)$.*
(E_7) *$f(x, y, z) = x^2 + y^3 + yz^3$ and $w_0 = (9, 6, 4)$.*
(E_8) *$f(x, y, z) = x^2 + y^3 + z^5$ and $w_0 = (15, 10, 6)$.*

Proof. Let $\widetilde{X} \to X$ and $\widetilde{Y}_1 \to Y_1$ be the minimal resolutions of X and Y_1, then there is a diagram

$$\begin{tikzcd} \widetilde{Y}_1 \arrow[r] \arrow[d] & \widetilde{X} \arrow[d] \\ Y_1 \arrow[r] & X. \end{tikzcd}$$

The pair (Y_1, F) has log terminal singularities, thus at F the singularities of (Y_1, F) are of the form $(\frac{1}{r}(1, a), (x = 0))$ [12, p. 119, Theorem 4.15]. The exceptional locus of $\widetilde{Y}_1 \to X$ consists of the strict transform F' of F together with strings of curves meeting F' obtained by resolution of the $(\frac{1}{r}(1, a), (x = 0))$ singularities of (Y_1, F). The exceptional locus of $\widetilde{X} \to X$ is either a string of rational curves or one of the D or E configurations. The map $\widetilde{Y}_1 \to \widetilde{X}$ is a composition of contractions of (-1)-curves.

Suppose that the exceptional locus of $\widetilde{Y}_1 \to X$ is a string of curves. In this case, the singularity $P \in X$ is a cyclic quotient singularity, and is of the form $\mathbb{A}^2_{u,v}/\frac{1}{kn^2}(1, kna - 1)$ by the classification of Theorem 2.4. Moreover, for an appropriate choice of the coordinates u, v, the morphism $\widetilde{Y}_1 \to X$ is toric. Then in particular p is toric and thus is a weighted blowup of u, v with weights given by some primitive vector in the lattice $\mathbb{Z}^2 + \mathbb{Z}\frac{1}{kn^2}(1, kna - 1)$.

Suppose now that the exceptional locus of $\widetilde{Y}_1 \to X$ is not a string of curves. We claim that $q : \widetilde{Y}_1 \to \widetilde{X}$ is an isomorphism. For, otherwise, F' must be a (-1)-curve, and q factors through the blow down $\sigma : \widetilde{Y}_1 \to Y_1'$ of F'. Then the exceptional locus of $Y_1' \to X$ has a point of multiplicity 3 or higher. On the other hand, the exceptional locus of $\widetilde{X} \to X$ has normal crossing singularities and the map $Y_1' \to \widetilde{X}$ is a composition of blowups. So the exceptional locus of $Y_1' \to X$ also has normal crossings, a contradiction. Hence q is an isomorphism, and the curve F' is the 'fork' curve of a D or E configuration (i.e., the exceptional curve which meets 3 others). This determines p uniquely, and we observe that these contractions can be described explicitly as above. □

Our main Theorem 2.5 follows immediately from Theorem 3.3 below.

Theorem 3.3. *Let $\pi : (E \subset \mathcal{Y}) \to (P \in \mathcal{X})/T$ be a semistable divisorial contraction with X normal. Let $p : (F \subset Y_1) \to (P \in X)$ be the induced log divisorial contraction. There is an isomorphism*

$$P \in X \cong \big(f(x, y, z) = 0\big) \subset \frac{1}{n}(1, -1, a)$$

such that p is given by the weighted blowup of x, y, z with weights w_0 as in Theorem 3.2—fix one such identification. Then there is a compatible isomorphism

$$P \in \mathcal{X}/T \cong \left(\big(f(x, y, z) + tg(x, y, z, t) = 0\big) \subset \frac{1}{n}(1, -1, a, 0)\right)\Big/\mathbb{C}^1_t$$

such that π is given by the weighted blowup of x, y, z, t with weights $w = (w_0, 1)$, where $w(tg) \geqslant w(f)$.

Conversely, let $\pi : \mathcal{Y} \to \mathcal{X}/T$ be a birational morphism constructed in this fashion. That is, $P \in \mathcal{X}/T \cong (f + tg = 0) \subset \frac{1}{n}(1, -1, a, 0)/\mathbb{C}^1_t$ and π is given by the weighted blowup of x, y, z, t with weights $w = (w_0, 1)$, where $(f(x, y, z) = 0) \subset \frac{1}{n}(1, -1, a)$ and w_0 are as in Theorem 3.2, and g is chosen so that $\mathcal{X}$ has an isolated singularity and $w(tg) \geqslant w(f)$. Then the family $\mathcal{Y}/T$ is semistable terminal, the divisor $-K_{\mathcal{Y}}$ is π-ample and the exceptional locus of π is an irreducible divisor.

Remark 3.4. Note that a morphism $\pi : \mathcal{Y} \to \mathcal{X}/T$ constructed as in the theorem is a semistable divisorial contraction precisely when $\rho(\mathcal{Y}/\mathcal{X}) = 1$. This is not always the case, e.g., if $\mathcal{X} = (xy + z^2 + t^2 = 0) \subset \mathbb{C}^4$ and $\pi : \mathcal{Y} \to \mathcal{X}$ is the blowup of $0 \in \mathcal{X}$ then $\rho(\mathcal{Y}/\mathcal{X}) = 2$, and π can be factored into a divisorial contraction followed by a flopping contraction. However, if $\mathcal{X}$ is $\mathbb{Q}$-factorial, then the exact sequence

$$0 \to \mathbb{Z}E \to \mathrm{Cl}(\mathcal{Y}) \to \mathrm{Cl}(\mathcal{X}) \to 0$$

implies that $\rho(\mathcal{Y}/\mathcal{X}) = 1$.

Note also that, in order to compute $\rho(\mathcal{Y}/\mathcal{X})$, we must work algebraically rather than local analytically at $P \in \mathcal{X}$. For example, we can construct a variety $\mathcal{X}$ which has an ordinary double point $P \in \mathcal{X}$ but is (algebraically) $\mathbb{Q}$-factorial. Then the blowup $\mathcal{Y} \to \mathcal{X}$ of $P \in \mathcal{X}$ has $\rho(\mathcal{Y}/\mathcal{X}) = 1$.

Lemma 3.5. *Let $\pi : (E \subset \mathcal{Y}) \to (P \in \mathcal{X})/T$ be a semistable divisorial contraction. Let $p : \tilde{\mathcal{X}} \to \mathcal{X}$ be the index one cover of $\mathcal{X}$. Then there is a diagram*

$$\begin{array}{ccc} \widetilde{E} \subset \widetilde{\mathcal{Y}} & \xrightarrow{\tilde{\pi}} & \widetilde{P} \in \widetilde{\mathcal{X}} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ E \subset \mathcal{Y} & \xrightarrow{\pi} & P \in \mathcal{X} \end{array}$$

where the map $q : \widetilde{\mathcal{Y}} \to \mathcal{Y}$ is a cyclic quotient, and the map $\tilde{\pi} : \widetilde{\mathcal{Y}} \to \widetilde{\mathcal{X}}$ is a birational morphism such that $\widetilde{\mathcal{Y}}$ is log terminal, the divisor $-K_{\widetilde{\mathcal{Y}}}$ is $\tilde{\pi}$-ample and the exceptional locus $\widetilde{E}$ of $\tilde{\pi}$ is an irreducible divisor.

Proof. Note first that $\widetilde{\mathcal{Y}}$ is the normalisation of the fibre product $\mathcal{Y} \times_{\mathcal{X}} \widetilde{\mathcal{X}}$. We give an explicit construction of $\widetilde{\mathcal{Y}}$ below and verify the desired properties.

Let a be the discrepancy of E and n the index of $P \in \mathcal{X}$. Then the index one cover of $P \in \mathcal{X}$ is given by

$$\widetilde{\mathcal{X}} = \underline{\mathrm{Spec}}_{\mathcal{X}}\left(\bigoplus_{i=0}^{n-1} \mathcal{O}_{\mathcal{X}}(iK_{\mathcal{X}})\right),$$

where the multiplication is defined by fixing an isomorphism $\mathcal{O}_{\mathcal{X}}(nK_{\mathcal{X}}) \cong \mathcal{O}_{\mathcal{X}}$. Explicitly, writing $P \in \mathcal{X}/T \cong ((f + tg = 0) \subset \frac{1}{n}(1, -1, a, 0))/\mathbb{C}^1_t$ as in Theorem 3.3, the index one cover is given by $((f + tg = 0) \subset \mathbb{C}^4)/\mathbb{C}^1_t$. Define $\widetilde{\mathcal{Y}}$ as follows:

$$\widetilde{\mathcal{Y}} = \underline{\mathrm{Spec}}_{\mathcal{Y}}\left(\bigoplus_{i=0}^{n-1} \mathcal{O}_{\mathcal{Y}}\big(iK_{\mathcal{Y}} + \lfloor -iaE \rfloor\big)\right),$$

where the multiplication is given by $\mathcal{O}_{\mathcal{Y}}(nK_{\mathcal{Y}} - naE) = \mathcal{O}_{\mathcal{Y}}(\pi^\star nK_{\mathcal{X}}) \cong \mathcal{O}_{\mathcal{Y}}$. The map $\tilde{\pi} : \widetilde{\mathcal{Y}} \to \widetilde{\mathcal{X}}$ is given by the natural maps

$$\mathcal{O}_{\mathcal{X}}(iK_{\mathcal{X}}) \to \pi_\star \mathcal{O}_{\mathcal{Y}}\big(iK_{\mathcal{Y}} + \lfloor -iaE \rfloor\big),$$

where we are using $\pi^\star K_{\mathcal{X}} \equiv K_{\mathcal{Y}} - aE$.

Write $a = a_1/d$ where a_1 and d are coprime integers. Then d divides n; let $e = n/d$. The exact sequence of groups

$$0 \to \mu_d \to \mu_n \to \mu_e \to 0$$

induces a factorisation $\widetilde{\mathcal{Y}} \to \mathcal{Y}' \to \mathcal{Y}$ where $\mathcal{Y}' \to \mathcal{Y}$ is etale in codimension 1: we have

$$\mathcal{Y}' = \widetilde{\mathcal{Y}}/\mu_d = \underline{\mathrm{Spec}}_{\mathcal{Y}}\left(\bigoplus_{j=0}^{e-1} \mathcal{O}_{\mathcal{Y}}(jdK_{\mathcal{Y}} - ja_1E)\right) \quad \text{and}$$
$$\widetilde{\mathcal{Y}} = \underline{\mathrm{Spec}}_{\mathcal{Y}'}\left(\bigoplus_{k=0}^{d-1} \mathcal{O}_{\mathcal{Y}'}\bigl(kK_{\mathcal{Y}'} + \lfloor -kaE' \rfloor\bigr)\right).$$

To understand the cover $\widetilde{\mathcal{Y}} \to \mathcal{Y}'$, choose b such that $-ba_1 = 1 \bmod d$ and, working locally over a point of $E' \subset \mathcal{Y}'$ where $\mathcal{Y}'$ is smooth, let v generate $\mathcal{O}_{\mathcal{Y}'}(bK_{\mathcal{Y}'} + \lfloor -baE' \rfloor)$. Then $\mathcal{O}_{\widetilde{\mathcal{Y}}}$ is freely generated locally by $1, v, \ldots, v^{d-1}$ over $\mathcal{O}_{\mathcal{Y}'}$ and $v^d = u \in \mathcal{O}_{\mathcal{Y}'}$ cuts out E'. In other words,

$$\widetilde{\mathcal{Y}} = \mathrm{Spec}\, \mathcal{O}_{\mathcal{Y}'}[s]/\bigl(s^d = u\bigr)$$

where $E' = (u = 0)$. So $\widetilde{\mathcal{Y}}$ is normal and $\widetilde{\mathcal{Y}} \to \mathcal{Y}'$ is totally ramified over E'.

We verify that $\widetilde{\mathcal{Y}}$ is log terminal. The pair $(\mathcal{Y}, Y)$ is dlt by the semistability assumption, in particular $(\mathcal{Y}, E)$ is log terminal. We have $K_{\widetilde{\mathcal{Y}}} = q^\star K_{\mathcal{Y}} + (d-1)\widetilde{E}$ by Riemann–Hurwitz and $q^\star E = d\widetilde{E}$, so $K_{\widetilde{\mathcal{Y}}} + \widetilde{E} = q^\star(K_{\mathcal{Y}} + E)$. Thus $(\mathcal{Y}, E)$ log terminal implies $(\widetilde{\mathcal{Y}}, \widetilde{E})$ log terminal and so $\widetilde{\mathcal{Y}}$ is log terminal as required. The remaining assertions are clear. $\square$

Remark 3.6. The contraction $\tilde{\pi}$ covering π is usually not a Mori contraction since $\widetilde{\mathcal{Y}}$ has log terminal but not terminal singularities. However, in some cases, $\tilde{\pi}$ is a Mori contraction. For example, let

$$\mathcal{X} = \bigl(xy + z^2 + t^N = 0\bigr) \subset \frac{1}{2}(1, 1, 1, 0), \quad N \geqslant 3,$$

a μ_2 quotient of a cA_1 point, and let π be given by the weighted blowup with weights $\frac{1}{2}(1, 5, 3, 2)$. Then $\widetilde{\mathcal{X}} = (xy + z^2 + t^N = 0) \subset \mathbb{A}^4$ and $\tilde{\pi}$ is the contraction given by the weighted blowup with weights $(1, 5, 3, 2)$. A calculation shows that $\tilde{\pi}$ is a Mori contraction precisely when $N = 3$. Moreover this is the only non-semistable contraction to a cA_1 point [9].

Proof of Theorem 3.3. A divisorial contraction $\pi : E \subset Y \to P \in X$ determines a discrete valuation $\nu : k(X)^\times \to \mathbb{Z}$, where $\nu(f)$ equals the order of vanishing of $\pi^\star f$ along E. Moreover, we can reconstruct π from ν. For, writing

$$m_{X,P}^{(n)} = \bigl\{ f \in \mathcal{O}_{X,P} \mid \nu(f) \geqslant n \bigr\},$$

we have $m_{X,P}^{(n)} = \pi_\star \mathcal{O}_Y(-nE)$ and

$$Y = \underline{\mathrm{Proj}}_X\left(\bigoplus_{n \geqslant 0} \pi_\star \mathcal{O}_Y(-nE)\right)$$

since $-E$ is π-ample. Hence, in order to identify two divisorial contractions it is enough to identify the corresponding valuations.

Consider the divisorial contraction $p: Y_1 \to X$; let $\nu_0 : k(X)^\times \to \mathbb{Z}$ be the corresponding valuation. Write $w_0 = \frac{1}{d}(a_1, a_2, a_3)$, where a_1, a_2, and a_3 are coprime. We have an inclusion

$$X = \bigl(f(x, y, z) = 0\bigr) \subset \mathbb{A} = \frac{1}{n}(1, -1, a)$$

and a weighted blowup

$$s : \mathrm{Bl}_{\frac{1}{d}(a_1,a_2,a_3)} \mathbb{A} \to \mathbb{A}$$

inducing the contraction p. Write $w_0 : k(\mathbb{A})^\times \to \mathbb{Z}$ for the valuation corresponding to the weighted blowup; for $h = \sum a_{ijk} x^i y^j z^k \in \mathcal{O}_{\mathbb{A},0}$ we have

$$w_0(h) = \min\left\{ \frac{1}{d}(a_1 i + a_2 j + a_3 k) \;\middle|\; a_{ijk} \neq 0 \right\}.$$

Then, for $h \in \mathcal{O}_{X,P}$,

$$\nu_0(h) = \max\bigl\{ w_0\bigl(\tilde{h}\bigr) \bigm| \tilde{h} \in \mathcal{O}_{\mathbb{A},0} \text{ a lift of } h \bigr\}.$$

More precisely, given a lift $\tilde{h}$ of h, write

$$\tilde{h} = \tilde{h}_\alpha + \tilde{h}_{\alpha+1} + \cdots, \quad \tilde{h}_\alpha \neq 0,$$

for the decomposition of $\tilde{h}$ into graded pieces with respect to the weighting. Then $\nu_0(h) = w_0(\tilde{h})$ iff $f \nmid \tilde{h}_\alpha$. For, the exceptional divisor G of s is a weighted projective space $\mathbb{P}(a_1, a_2, a_3)$, and the exceptional divisor F of p is given by

$$F = \bigl(f(X, Y, Z) = 0\bigr) \subset \mathbb{P}(a_1, a_2, a_3)$$

(note that f is homogeneous with respect to the weighting). Now, $s^\star \tilde{h} = u^\alpha \tilde{h}'$, where u is a local parameter at G, and $(\tilde{h}' = 0)$ is the strict transform of $(\tilde{h} = 0) \subset \mathbb{A}$. Thus $p^\star h = s^\star \tilde{h}|_{Y_1} = v^\alpha \cdot \tilde{h}'|_{Y_1}$, where v is a local parameter at F. So $\nu_0(h) \geqslant \alpha = w_0(\tilde{h})$, with equality iff $\tilde{h}'|_{Y_1}$ does not vanish on F. We have

$$\bigl(\tilde{h}' = 0\bigr)|_G = \bigl(\tilde{h}_\alpha(X, Y, Z) = 0\bigr) \subset \mathbb{P}(a_1, a_2, a_3),$$

so this last condition is equivalent to $f \nmid \tilde{h}_\alpha$.

Our aim is to deduce a description of the contraction $\pi : \mathcal{Y} \to \mathcal{X}$ similar to that of $p: Y_1 \to X$ above. Let ν be the valuation defined by π. We first show that, in the case that the index n of $P \in \mathcal{X}$ equals 1, we may lift $x, y, z \in \mathcal{O}_{X,P}$ to $\tilde{x}, \tilde{y}, \tilde{z} \in \mathcal{O}_{\mathcal{X},P}$ such that

$\nu(\tilde{x}) = \nu_0(x)$, etc. Certainly, for $h \in \mathcal{O}_{X,P}$ and any lift $\tilde{h} \in \mathcal{O}_{\mathcal{X},P}$, we have $\nu(\tilde{h}) \leqslant \nu_0(h)$; we show that for an appropriate choice of $\tilde{h}$ we have equality. Equivalently, the map

$$\pi_\star \mathcal{O}_\mathcal{Y}(-iE) \to p_\star \mathcal{O}_{Y_1}(-iE)$$

is surjective for each $i \geqslant 1$. Applying $\pi_\star$ to the exact sequence

$$0 \to \mathcal{O}_\mathcal{Y}(-iE - Y_1) \to \mathcal{O}_\mathcal{Y}(-iE) \to \mathcal{O}_{Y_1}(-iE) \to 0$$

we obtain the long exact sequence

$$\cdots \to \pi_\star \mathcal{O}_\mathcal{Y}(-iE) \to p_\star \mathcal{O}_{Y_1}(-iE) \to R^1\pi_\star \mathcal{O}_\mathcal{Y}(-iE - Y_1) \to \cdots.$$

We have $Y_1 + E = Y \sim 0$ and $K_\mathcal{Y} \sim \pi^\star K_\mathcal{X} + aE \sim aE$, so $-iE - Y_1 \sim K_\mathcal{Y} - (i-1+a)E$. Thus $R^1\pi_\star \mathcal{O}_\mathcal{Y}(-iE - Y_1) = R^1\pi_\star \mathcal{O}_\mathcal{Y}(K_\mathcal{Y} - (i-1+a)E) = 0$ by Kodaira vanishing since $-E$ is π-ample; the required surjectivity result follows. In the case that the index is greater than 1, by Lemma 3.5 there is a diagram

$$\begin{tikzcd} \widetilde{E} \subset \widetilde{\mathcal{Y}} \arrow[r, "\tilde{\pi}"] \arrow[d] & \widetilde{P} \in \widetilde{\mathcal{X}} \arrow[d] \\ E \subset \mathcal{Y} \arrow[r, "\pi"] & P \in \mathcal{X} \end{tikzcd}$$

where $\widetilde{\mathcal{X}} \to \mathcal{X}$ is the index one cover. Then a similar calculation shows that we may lift $x, y, z \in \mathcal{O}_{\widetilde{X},P}$ to $\tilde{x}, \tilde{y}, \tilde{z} \in \mathcal{O}_{\widetilde{\mathcal{X}},P}$ of the same weights as above.

For simplicity of notation, now write x, y, z for the lifts $\tilde{x}, \tilde{y}, \tilde{z} \in \mathcal{O}_{\widetilde{\mathcal{X}},P}$ of $x, y, z \in \mathcal{O}_{\widetilde{X},P}$ constructed above. Let t denote a local parameter at $0 \in T$. Then x, y, z, t define an embedding $\mathcal{X} \hookrightarrow \mathbb{A} \times T$ extending $X \hookrightarrow \mathbb{A}$; write

$$\mathcal{X} = \big(f(x, y, z) + tg(x, y, z, t) = 0\big) \subset \mathbb{A} \times T = \frac{1}{n}(1, -1, a, 0).$$

Define a weighted blowup

$$\sigma : \mathrm{Bl}_{\frac{1}{d}(a_1,a_2,a_3,d)}(\mathbb{A} \times T) \to \mathbb{A} \times T,$$

and let w denote the corresponding valuation. Our aim is to show that π is induced by σ. Note immediately that, for $h \in \mathcal{O}_{\mathcal{X},P}$, and any lift $\tilde{h} \in \mathcal{O}_{\mathbb{A}\times T,0}$, we have $\nu(h) \geqslant w(\tilde{h})$. For, the valuations ν and w agree on x, y and z by construction, and $\nu(t) = 1 = w(t)$ by the semistability assumption. So, writing $\tilde{h} = \sum a_{ijkl} x^i y^j z^k t^l$, we have

$$\nu(h) \geqslant \min\big\{\nu\big(x^i y^j z^k t^l\big) \bigm| a_{ijkl} \neq 0\big\} = \min\big\{w\big(x^i y^j z^k t^l\big) \bigm| a_{ijkl} \neq 0\big\} = w\big(\tilde{h}\big)$$

as claimed. We also need the following preliminary result: $w(tg) \geqslant w(f)$. To prove this, write

$$g = g_\alpha + g_{\alpha+1} + \cdots, \quad g_\alpha \neq 0$$

for the decomposition of g into graded pieces with respect to the weighting. Let $g_\alpha = t^\beta k$, where $k_0 = k|_{t=0} \neq 0$. We may assume that f does not divide k_0, for otherwise $w(tg) > w(f)$. In this case, we have $\nu_0(k_0) = w_0(k_0)$ as proved above. Now $w(k) \leqslant \nu(k) \leqslant \nu_0(k_0) = w_0(k_0) = w(k)$, thus $w(k) = \nu(k)$ and so $w(g) = \nu(g)$. Finally $w(f) \leqslant \nu(f) = \nu(tg) = w(tg)$ as required.

We now complete the proof that π is induced by σ. The contraction $\pi' : E' \subset \mathcal{Y}' \to P \in \mathcal{X}$ induced by σ has valuation ν' given by

$$\nu'(h) = \max\bigl\{w(\tilde{h}) \bigm| \tilde{h} \in \mathcal{O}_{\mathbb{A}\times T,0} \text{ a lift of } h\bigr\}$$

(cf. our earlier treatment of the contraction p induced by s). We show that $\nu = \nu'$ and thus $\pi = \pi'$ as required. We know that $\nu(h) \geqslant w(\tilde{h})$ for $h \in \mathcal{O}_{\mathcal{X},P}$ and $\tilde{h} \in \mathcal{O}_{\mathbb{A}\times T,0}$ any lift of h, it remains to show that we have equality for some $\tilde{h}$. Given $h \in \mathcal{O}_{\mathcal{X},P}$, pick some lifting $\tilde{h}$. Write

$$\tilde{h} = \tilde{h}_\alpha + \tilde{h}_{\alpha+1} + \cdots$$

for the decomposition into graded pieces with respect to the weighting and $\tilde{h}_\alpha = t^\beta k$, where $k_0 = k|_{t=0} \neq 0$. If f divides k_0, say $k_0 = q \cdot f$, replace $\tilde{h}$ by $\tilde{h}' = \tilde{h} - q(f + tg)$. Defining α' and β' as above, we see that either $\alpha' > \alpha$ or $\alpha' = \alpha$ and $\beta' > \beta$, using $w(tg) \geqslant w(f)$. It follows that this process can repeat only finitely many times, so we may assume $f \nmid k_0$ and hence $\nu_0(k_0) = w_0(k_0)$. Then $w(k) \leqslant \nu(k) \leqslant \nu_0(k_0) = w_0(k_0) = w(k)$, so $w(k) = \nu(k)$ and $w(\tilde{h}) = \nu(h)$ as required.

Conversely, we show that all contractions $\pi : \mathcal{Y} \to \mathcal{X}$ constructed in this way have all the properties of a semistable divisorial contraction, except that the relative Picard number is not necessarily equal to 1. We first prove that $(\mathcal{Y}, Y)$ is dlt. We know that the pair (Y_1, F) is log terminal and

$$E = \bigl(f(X, Y, Z) + Tg_{\lambda-1}(X, Y, Z, T) = 0\bigr) \subset \mathbb{P}(a_1, a_2, a_3, d),$$

where $\lambda = w(f)$ and $g_{\lambda-1}$ is the graded piece of g with weight $\lambda - 1$ (possibly zero). The curve $F = (T = 0) \subset E$ is smooth and the only singularities of E at F are cyclic quotient singularities induced by the singularities of the ambient weighted projective space. Hence the pair (E, F) is log terminal near F. But $-(K_E + F)$ is ample, so (E, F) is log terminal everywhere by Shokurov's connectedness result (see Lemma 3.7 below). Hence the degenerate surface $Y = Y_1 + E$ is semi log terminal [13]. A standard inversion of adjunction argument shows that $(\mathcal{Y}, Y)$ is dlt. The divisor E is clearly irreducible. Finally, $-K_{\mathcal{Y}}$ is π-ample. For $K_{\mathcal{Y}} \equiv aE$ and $a > 0$ since $\mathcal{X}$ is terminal. □

Lemma 3.7. *Let X be a normal proper surface and B an effective $\mathbb{Q}$-divisor on X such that $-(K_X + B)$ is nef and big. Then the locus where (X, B) is not klt is connected.*

Proof. This is a global version of a connectedness result due to Shokurov and Kollár (see, e.g., [12, p. 173, Theorem 5.48])—the proof is essentially unchanged but is included below for completeness.

Let $f : Y \to X$ be a resolution of X such that the support of the strict transform of B together with the exceptional locus of f form a snc divisor on Y. Define D via the equation

$$K_Y + D = f^\star(K_X + B)$$

and let $D = \sum d_i D_i$ where the D_i are the irreducible components of $\operatorname{Supp} D$. Decompose D into $A = \sum_{d_i<1} d_i D_i$ and $F = \sum_{d_i \geqslant 1} d_i D_i$. Then the locus where (X, B) is not klt is the image of F, so it is enough to show that $\operatorname{Supp} F$ is connected. By definition we have

$$-A - F = -D = K_Y - f^\star(K_X + B).$$

Rounding up we obtain

$$\lceil -A \rceil - \lfloor F \rfloor = K_Y - f^\star(K_X + B) + \{A\} + \{F\}.$$

Since $-f^\star(K_X + B)$ is nef and big and $\{A\} + \{F\}$ has snc support and coefficients smaller than 1, we deduce that $H^1(\mathcal{O}_Y(\lceil -A \rceil - \lfloor F \rfloor)) = 0$ by Kodaira vanishing. Hence the map

$$H^0\big(\mathcal{O}_Y\big(\lceil -A \rceil\big)\big) \to H^0\big(\mathcal{O}_{\lfloor F \rfloor}\big(\lceil -A \rceil\big)\big)$$

is surjective. The divisor $\lceil -A \rceil$ is effective and f-exceptional. So

$$H^0\big(\mathcal{O}_Y\big(\lceil -A \rceil\big)\big) = H^0(\mathcal{O}_X) \cong \mathbb{C}$$

and $H^0(\mathcal{O}_{F_j}(\lceil -A \rceil)) \neq 0$ for any connected component F_j of $\lfloor F \rfloor$. Thus $\operatorname{Supp}\lfloor F \rfloor = \operatorname{Supp} F$ is connected by the surjectivity result above, as required. □

4. Examples

We give a detailed description of the semistable contractions

$$\pi : (E \subset \mathcal{Y}) \to (P \in \mathcal{X})/T$$

where $P \in \mathcal{X}/T$ is of the form

$$\big(xy + z^{kn} + tg(x, y, z, t) = 0\big) \subset \frac{1}{n}(1, -1, a, 0)/\mathbb{C}^1_t.$$

By our theorem we may choose the coordinates x, y, z, t so that the contraction π is given by the weighted blowup of x, y, z, t with weights $w = (w_0, 1)$, where $w_0 \in \mathbb{Z}^3 + \mathbb{Z}\frac{1}{n}(1, -1, a)$ is a primitive vector, $f(x, y, z) = xy + z^{kn}$ is homogeneous with respect to w_0 and $w(tg) \geqslant w(f)$.

Example 4.1. Consider first the case where the index n equals 1. Let $w_0 = (a_1, a_2, a_3)$. After a change of coordinates we may assume that

$$tg(x, y, z, t) = b_{k-2}z^{k-2}t^{2a_3} + b_{k-3}z^{k-3}t^{3a_3} + \cdots + b_0t^{ka_3},$$

where $b_i \in k[[t]]$ for each i, using $w(tg) \geqslant w(f)$. Then the exceptional divisor E of π is the surface

$$E = \bigl(XY + Z^k + c_{k-2}Z^{k-2}T^{2a_3} + \cdots + c_0T^{ka_3} = 0\bigr) \subset \mathbb{P}(a_1, a_2, a_3, 1)$$

where $c_i = b_i(0) \in k$, and X, Y, Z, T are the homogeneous coordinates on $\mathbb{P}(a_1, a_2, a_3, 1)$ corresponding to the coordinates x, y, z, t on $\mathbb{A}^4$.

We now describe the singularities of the pair $(\mathcal{Y}, Y)$. Consider first the affine piece $U = (T \neq 0)$ of E:

$$U = \bigl(x'y' + z'^k + c_{k-2}z'^{k-2} + \cdots + c_0 = 0\bigr) \subset \mathbb{A}^3.$$

The only singularities of U are A_{l-1} singularities at the points where $x' = y' = 0$ and $h(z') = z'^k + c_{k-2}z'^{k-2} + \cdots + c_0$ has a multiple root of multiplicity l. These singularities are sections of cA_{l-1} singularities on the total space $\mathcal{Y}$. The remaining singularities of $Y \subset \mathcal{Y}$ are of the form $(xy = 0) \subset \frac{1}{a_1}(1, -1, a_3)$ and $(xy = 0) \subset \frac{1}{a_2}(1, -1, a_3)$ at the points $(1:0:0:0)$ and $(0:1:0:0)$ of E respectively.

Example 4.2. Let the index $n \in \mathbb{N}$ be arbitrary. Let $w_0 = \frac{1}{d}(a_1, a_2, a_3)$, where a_1, a_2, and a_3 are coprime, and write $n = de$. We may assume that

$$tg(x, y, z, t) = b_{k-2}z^{(k-2)n}t^{2ea_3} + b_{k-3}z^{(k-3)n}t^{3ea_3} + \cdots + b_0t^{kea_3},$$

where $b_i \in k[[t]]$ for each i, and we have used the fact that g is μ_n invariant. The exceptional divisor E of π is the surface

$$E = \bigl(XY + Z^{kn} + c_{k-2}Z^{(k-2)n}T^{2ea_3} + \cdots + c_0T^{kea_3} = 0\bigr) \subset \mathbb{P}(a_1, a_2, a_3, d)$$

where $c_i = b_i(0) \in k$. Consider the affine piece $U = (T \neq 0)$ of E:

$$U = \bigl(x'y' + z'^{kn} + c_{k-2}z'^{(k-2)n} + \cdots + c_0 = 0\bigr) \subset \frac{1}{d}(a_1, a_2, a_3).$$

The singularities of U away from $(0, 0, 0)$ can be described as above. At $(0, 0, 0)$ we have a singularity of the form

$$\bigl(xy + z^{ln} = 0\bigr) \subset \frac{1}{d}(a_1, a_2, a_3) \cong \frac{1}{d}(1, -1, b)$$

where l is the least i such that $b_i \neq 0$ and $b \equiv a_1^{-1} a_3 \bmod d$. This is a cyclic quotient singularity of the form $\frac{1}{k'n'^2}(1, k'n'a' - 1)$, where $k' = le$, $n' = d$ and $a' = b$. The corresponding singularity of the total space $\mathcal{Y}$ is a deformation of $0 \in U$ of the form

$$\bigl(xy + z^{ln} + tg\bigl(z^d, t\bigr) = 0\bigr) \subset \frac{1}{d}(1, -1, b, 0).$$

The remaining singularities of $Y \subset \mathcal{Y}$ are of the form $(xy = 0) \subset \frac{1}{ea_1}(1, -1, \frac{a_3 - aa_1}{d})$ and $(xy = 0) \subset \frac{1}{ea_2}(1, -1, \frac{aa_2 + a_3}{d})$ at the points $(1 : 0 : 0 : 0)$ and $(0 : 1 : 0 : 0)$ of E respectively.

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